deflection Δ of point B, the total force transmitted to member AB by member BC consists of a vertical component P and a transverse component $R = P\Delta/cL$. Thus, for decreasing values of c, member AB is subjected to a vertical force P in addition to an increasing transversal force varying inversely with c. This therefore results in a decreasing buckling load with decreasing length cL.

As for the behavior when $c \to 0$, the solution curves given in Fig. 2 of the original paper are valid for all $0<c$. It was pointed out originally that when $c=0$ identically, the buckling load is in fact that of a fixed-hinged member [see Eq. (6)] and not zero. This was indicated most clearly by demonstrating that the relevant root of the transcendental equation is given by point A of Fig. 3, when $c=0$. Dr. Johns has failed to notice this point, since his dicussion is based on the implication that the solution curves of Fig. 2 are meant to include the value of $c=0$.

It is a well-known phenomenon in structural mechanics that, in a general solution, when two hinges are made to coincide (here $c=0$), a singular behavior usually will occur, and in the paper a mathematical explanation of this isolated case was given. In no sense, then, did the meaning of the paradox apply to this singular case.

Be means of his example, Dr. Johns confirms that it is possible, and clearly not in an imaginary sense, to obtain increasing instability loads for members of increasing length with values $c>0$. Although for two-dimensional plate problems it is known that increasing a dimension can result in increased stability,[1] it is rather unusual, if not paradoxical, that for a one-dimensional case the same phenomenon occurs when a member is subjected to axial compressive loads.

References

[1] Timoshenko, S. and Gere, T., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961, pp. 351 ff.

Comment on "A Paradoxical Case in a Stability Analysis"

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PARNES[1] states with respect to column members that "it is generally accepted that the load at which instability occurs decreases as the member increases in length." He terms "paradoxical" a specific model that he presents in which the opposite is true. However, even in the standard text on elastic stability by Timoshenko and Gere,[2] models of such "paradoxical" elastic systems are presented. In addition, structural models having such properties have been used to study nonlinear buckling phenomena by Panovko and Gubanova.[3]

The explanation of the supposed paradox is that, if concentrated elastic restraint against displacement occurs at the ends of a rigid or relatively rigid member that is axially loaded, the rotational stiffness of the member is the one pertinent to elastic stability, and this increases with length of the member. One system considered by Timoshenko and Gere, shown in Fig. 1, consists of an infinitely rigid bar of length ℓ hinged at one end and laterally restrained by a spring of spring constant K at the other end. The axial load at which buckling occurs is given as

$$P = K\ell \tag{1}$$

which is the condition for neutral equilibrium of horizontal forces when the bar undergoes a small angular rotation α. Other more complex cases of pin-connected rigid bars under lateral restraints are also given by Timoshenko and Gere and also lead to buckling loads that increase with length.

In Parnes' first model, shown in Fig. 2, the linear spring of Fig. 1 is effectively replaced by a cantilever beam-column member AB of stiffness EI and length L, restraining the lateral motion of the rigid link BC, of length CL and also carrying though the same compressive load P as imposed on the rigid link. The formula for lateral tip deflection of a cantilever beam-column, y_t, under lateral force F and axial load P, is[4]

$$y_t = \frac{FL}{P}\left(\frac{\tan\beta L}{\beta L} - 1\right) \tag{2}$$

where $\beta^2 = P/EI$, so that the stiffness $K(P)$ is

$$K(P) = F/y_t = P/L\left(\frac{1}{\tan\beta L/\beta L - 1}\right) \tag{3}$$

For small values of P/EI (i.e., well below the buckling value of $\pi^2/4L^2$),

$$\frac{\tan\beta L}{\beta L} - 1 \approx 1 + \frac{(\beta L)^2}{3} - 1 \approx \frac{(\beta L)^2}{3} \tag{4}$$

or

$$K(P) \approx \frac{P}{L}\,\frac{1}{(PL^4/3EI)} = \frac{3EI}{L^3} \tag{5}$$

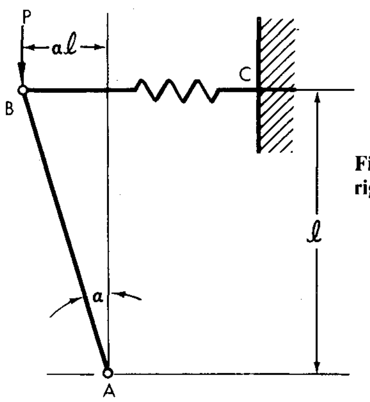

Fig. 1 Stability problem with rigid links.

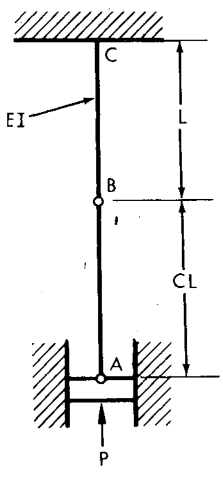

Fig. 2 Stability problem with flexible links.

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This will be recognized as being equal to the lateral stiffness of a beam without axial load, since the tip deflection y of such a beam under lateral load F at the tip is given by

$$y_t = FL^3/3EI \tag{6}$$

Using the beam-column stiffness $K(P)$ in place of K in the Timoshenko and Gere formula, Eq. (1) gives

$$P = \frac{P(CL)}{L} \frac{1}{(\tan\beta L/\beta L - 1)} \tag{7}$$

or

$$(\tan\beta L/\beta L) - 1 = C \tag{8}$$

which is exactly Parnes' Eq. (4). For small values of C, we have the buckling of the rigid link in rotation as the predominating failure mode. From Eqs. (1) and (5) of this Comment or by use of the series expansion Eq. (4) in Eq. (8), this load is

$$P \approx (3EI/L^3)CL \tag{9}$$

This illustrates the "paradoxical" behavior at small values of C noted by Parnes and is seen here to be essentially the same physical phenomenon dealt with by Timoshenko and Gere.

As C approaches infinity, buckling of the elastic column predominates, and from Eq. (8) the critical load is given by $\beta L = \pi/2$, which is the classical buckling load of the cantilever column. As Parnes showed, consideration of the finite stiffness of the link BC introduces no interaction with the phenomena just discussed but merely requires the additional consideration of a possible buckling condition of the link BC, considered to be simply supported at both ends.

References

[1] Parnes, R., "A Paradoxical Case in a Stability Analysis," *AIAA Journal*, Vol. 15, Oct. 1977, pp. 1533-1534.

[2] Timoshenko, S. P. and Gere, J. M., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961, pp. 83-86.

[3] Panovko, Y. G. and Gubanova, I. I., "Stability and Oscillation of Elastic Systems," Consultants Bureau, New York, 1965, pp. 53-54.

[4] Roark, R. J., *Formulas For Stress and Strain*, 4th ed., McGraw-Hill, New York, 1965, p. 148.

Reply by Author to A. H. Flax

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THE author wishes to thank Dr. Flax for his pertinent discussion of the apparent "paradox" and for bringing to his attention an explanation of the paradox by considering the lateral deflection of a rigid link resisted by a spring of effective stiffness $K(P)$. While this explanation is quite correct and ingenious, the author believes the explanation given in the previous discussion[1] to be more direct, as it derives immediately from considerations of statics.

Although it is well known that systems of rigid links connected in various ways by means of elastic springs often lead to increasing values of critical loads with member length, the author believes that this phenomenon had not been previous shown to exist for cases of elastic columns that undergo flexure.

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[1] Parnes, R., "Reply by Author to D.J. Johns," *AIAA Journal*, Vol. 16, Sept. 1978, p. 1115.

Comment on "Effects of Radial Appendage Flexibility on Shaft Whirl Stability"

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WILGEN and Schlack[1] investigate the stability of a rotating shaft with two opposite radial booms attached to it (see Fig. 1). They give the stability limits as a function of the flexibilities of the components and of other system parameters. In the following it is shown that 1) analytical solutions can be given for the shaft deflections which are approximated by a series in Ref. 1, 2) simple and accurate numerical solutions do exist for the deflections of the radial booms, which are approximated by one single "comparison function" in Ref. 1, and 3) the out-of-plane stability of the system is strongly dependent on the flexibility of the radial booms and not independent of that parameter as stated in Ref. 1. The in-plane stability limits given in Ref. 1 will be recalculated on the basis of the preceding comments and new stability limits will be given for the out-of-plane motion.

The existence of analytical solutions for rotating shafts is well known.[2] Less known perhaps is the fact that analytical solutions for rotating symmetric shafts can be derived from the well-known solutions for nonrotating shafts (i.e., booms) by pure coordinate transformation.[3] This is a direct consequence of the fact that a rotation with respect to the shaft axis is not reflected in the differential equations for a symmetric shaft if the system of reference is not affected by this rotation. In general, however (but not always), rotating coordinates are used with rotating shafts. That is why the spin appears in the equations. The fact that the differential equations (and consequently the dynamic behavior) of a rotating symmetric shaft are not affected by the speed of rotation is only true for conservative systems. Inner damping forces, for instance, do depend on the shaft's rotation, and it is only due to these forces that a shaft becomes unstable at a certain critical speed.

For radial booms, on the other hand, analytical solutions are not known because the underlying differential equations have nonconstant parameters arising from the nonconstant axial (centrifugal) force. However, numerical solutions of the differential equations can be given with a precision that depends only on computer accuracy and not on modal truncation or discretization effects. Here the boom deflections are expressed by a power series.

The recurrence law for the generation of the coefficients of the series follows directly from the coefficients of the dif-

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